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## Group-Theoretical Aspects of Instantons.

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(ricevuto il 23 Febbraio 1979)

**Summary.** — We discuss the problem of embeddings in non-Abelian gauge theories. The (ir)reducibility of a gauge field configuration is characterized. For the specific case of instanton solutions we derive a practical criterion for  $SU_n$ . In the general construction of self-dual solutions the reducibility of the gauge field implies certain symmetry conditions on the parameter space. We determine these conditions for  $Sp_n$ ,  $O_n$  and  $SU_p \times SU_q$  embeddings.

### 1. — Introduction.

The set of gauge field configurations of a gauge group  $G$  contains, as particular cases, gauge field configurations of subgroups of  $G$ . In certain applications it is important to know whether or not a gauge field corresponding to a group  $G$  is reducible to some smaller group included in  $G$ . In particular, problems of this kind have been encountered in the construction of instantons (solutions of the Euclidean equations of motion of pure Yang-Mills theories) for gauge groups other than  $SU_2$ . In this paper we shall use group-theoretical methods to study various aspects of the problem of embeddings of gauge groups. We shall review some of the work which has already been done in this field and at the same time present a number of new results.

The mathematical characterization of embeddings has been given by DYNKIN <sup>(1)</sup>. In sect. 2 we shall present aspects of his classification which are

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<sup>(1)</sup> E. DYNKIN: *Mat. Sb.*, **30**, 349 (1952), American Mathematical Society Translations (Providence, R. I., 1957), Ser. 2, Vol. **6**, p. 111.

relevant to the problem of embeddings of gauge fields. In particular, we shall see that the Dynkin index plays an important role in the calculation of topological charges. In sect. 3 we shall give a theorem which provides necessary and sufficient conditions to recognize embeddings <sup>(2)</sup>.

Sections 2 and 3 are general in the sense that the results given there apply to an arbitrary gauge field configuration. In the sections that follow we shall specialize to instanton solutions. Let us recall that until a few months ago the only explicitly known solutions of the Euclidean Yang-Mills equations were for the  $SU_2$ -group <sup>(3)</sup>. Several authors had written solutions for larger groups <sup>(4)</sup>, which were, intentionally or nonintentionally, embeddings of  $SU_2$  in these groups. Recently an irreducible  $SU_3$  solution was given <sup>(5)</sup>, and with criteria we had obtained previously <sup>(2)</sup> we could show that this solution was indeed irreducible. In sect. 4 we prove a similar criterion for  $SU_n$ , which we have used to show that a  $SU_n$  solution given in ref. <sup>(6)</sup> is also irreducible.

The construction of ATIYAH *et al.* <sup>(7)</sup> gives in principle all self-dual gauge fields for any compact Lie group. The remaining problem is to write a matrix  $M$  which has a certain rank, is positive Hermitian, and satisfies certain symmetry properties <sup>(8)</sup>. The most general matrix of this kind is not known explicitly, so that for any particular example one may still find solutions that belong to a smaller gauge group than the one originally considered. For some subgroups of  $SU_n$  we give conditions on the matrix  $M$  which characterize the reducibility of the gauge field. This shows how the instantons for the groups  $O_n$ ,  $Sp_n$  and  $SU_p \times SU_q$  appear in a natural way as special cases in the  $SU_n$  construction. These and other properties of the parameter space for  $SU_n$  instantons are discussed in sect. 5.

## 2. – Definition and general properties of embeddings.

The mathematical problem of embedding a simple algebra  $\tilde{G}$  in a simple algebra  $G$  has been discussed by DYNKIN <sup>(1)</sup>.

<sup>(2)</sup> C. MEYERS, M. DE ROO and P. SORBA: *Nucl. Phys.*, **140** B, 533 (1978).

<sup>(3)</sup> A. A. BELAVIN, A. M. POLYAKOV, A. S. SCHWARTZ and YU. S. TYUPKIN: *Phys. Lett.*, **59** B, 85 (1975); E. WITTEN: *Phys. Rev. Lett.*, **38**, 121 (1977); G. 'T HOOFT: unpublished; R. JACKIW, C. NOHL and C. REBBI: *Phys. Rev. D*, **15**, 1642 (1977).

<sup>(4)</sup> F. WILCZEK: *Phys. Lett.*, **65** B, 160 (1976); R. C. YATES: *Nucl. Phys.*, **123** B, 333 (1977); D. H. TCHRAKIAN and J. H. RAWNSLEY: *J. Phys. A*, **10**, L195 (1977); Y. BRIHAYE and J. NUYTS: *Phys. Lett.*, **66** B, 346 (1977).

<sup>(5)</sup> F. A. BAIS and H. A. WELDON: University of Pennsylvania preprint (1978).

<sup>(6)</sup> F. A. BAIS and H. A. WELDON: University of Pennsylvania preprint (July 1978).

<sup>(7)</sup> M. F. ATIYAH, N. J. HITCHIN, V. G. DRINFELD and YU. I. MANIN: *Phys. Lett.*, **65** A, 185 (1978).

<sup>(8)</sup> C. MEYERS and M. DE ROO: *Nucl. Phys.*, **148** B, 61 (1979).

A faithful embedding of an algebra  $\tilde{G}$  in an algebra  $G$  is defined by an injective mapping  $f$  of  $\tilde{G}$  into  $G$ :  $\tilde{X} \rightarrow f(\tilde{X}) \in G$  for every  $\tilde{X} \in \tilde{G}$  such that

$$(2.1) \quad f([\tilde{X}, \tilde{Y}]) = [f(\tilde{X}), f(\tilde{Y})].$$

Now let  $G$  be a semi-simple algebra and  $L$  its Cartan subalgebra; let  $\tilde{G}$  be a subalgebra of  $G$  with  $\tilde{K}$  its Cartan subalgebra. Then an embedding of  $\tilde{G}$  into  $G$  is completely defined by a mapping from  $\tilde{K}$  into  $K$ :

$$(2.2) \quad f(\tilde{H}_i) = \sum_{k=1}^n f_{ik} H_k, \quad i = 1, \dots, l,$$

where  $\tilde{H}_i$  and  $H_i$  are elements of  $\tilde{K}$  and  $K$ , respectively,  $l$  is the rank of  $\tilde{G}$  and  $n$  the rank of  $G$ .

Moreover, we have the relation

$$(2.3) \quad (f(\tilde{X}), f(\tilde{Y})) = j_f \cdot (\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in \tilde{G},$$

in which  $(\tilde{X}, \tilde{Y}) = \text{Tr ad } \tilde{X} \text{ ad } \tilde{Y}$  is the value of the Killing form for  $\tilde{X}$  and  $\tilde{Y}$ . This relation determines a scalar factor  $j_f$  independent of  $\tilde{X}, \tilde{Y}$  which is called the Dynkin index of the embedding. The matrix  $[f_{ik}]$  is called the defining matrix of the embedding. Obviously two equivalent embeddings have the same matrix  $[f_{ik}]$  and the same index  $j_f$  (two embeddings  $f_1$  and  $f_2$  of  $\tilde{G}$  into  $G$  are equivalent if the algebras  $f_1(\tilde{G})$  and  $f_2(\tilde{G})$  are conjugated by elements of  $G$ ). In most cases, the indices corresponding to inequivalent embeddings are different and hence may be used to label the embeddings<sup>(1,9)</sup>. When the indices of two inequivalent embeddings coincide, we must then refer to the defining matrices.

By using an orthonormal basis for  $\tilde{G}$  and  $G$ , eq. (2.3) is equivalent to

$$(2.4) \quad \sum_{k=1}^n f_{ik} f_{mk} = j_f \delta_{im}.$$

Note that in the case of  $\tilde{G} \simeq SU_2$ , the matrix  $[f_{ik}]$  becomes a vector  $[f_k]$ ,  $k = 1, \dots, n$ , called the defining vector of the embedding. In this case the index is given by

$$(2.5) \quad j_f = \sum_{k=1}^n f_k^2.$$

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<sup>(9)</sup> M. LORENTE and B. GRUBER: *Journ. Math. Phys.*, **13**, 1639 (1972); B. GRUBER and M. T. SAMUEL: in *Group Theory and its Applications*, Vol. **3**, edited by E. M. LOEBL (New York, N. Y., 1975).

There are, for example, two inequivalent embeddings of  $SU_2$  in  $SU_3$  with  $j_r = 1$  and  $j_r = 4$ , respectively.

Some of the properties of this index are the following:

- i)  $j_r$  is a nonnegative integer number.
- ii) If  $G_1 \subseteq G_2 \subseteq G_3$  are simple algebras, the index of  $G_1$  in  $G_2$  is the product of the indices of  $G_1$  in  $G_2$  and  $G_2$  in  $G_3$ :

$$j_{G_1/G} = j_{G_1/G_2} \cdot j_{G_2/G_3}.$$

- iii) If  $f_1, f_2, \dots, f_s$  are embeddings of a simple algebra  $\tilde{G}$  in the simple algebra  $G$  and if

$$[f_i(\tilde{X}), f_j(\tilde{Y})] = 0$$

for any couple of indices  $i \neq j$  and for any two elements  $\tilde{X}, \tilde{Y} \in \tilde{G}$ , then  $f = f_1 + f_2 + \dots + f_s$  is again an embedding and  $j_f = j_{f_1} + j_{f_2} + \dots + j_{f_s}$ .

Now let us define what we mean by an embedding of a gauge field. Let  $A_\mu$  be a gauge potential on the Euclidean space  $E^4$ , defined by analytic functions  $A_\mu^\alpha(x)$ ,

$$A_\mu(x) = A_\mu^\alpha(x) X_\alpha,$$

where  $X_\alpha$  are the generators of a compact gauge group  $G$ . Then the gauge potential  $A_\mu(x)$  of  $G$  is an embedding of  $H$  in  $G$  (or is reducible to  $H$ ) if there exists a gauge in which  $A_\mu(x)$  belongs to the algebra  $\mathcal{H}$  of the proper subgroup  $H$  of  $G$ .

If the gauge field  $A_\mu$  is an embedding of  $\tilde{G}$  in  $G$ , the Dynkin index of the embedding,  $j_{\tilde{G}/G}$ , plays an important role in the calculation of the topological charge. The charge will be

$$(2.6) \quad q = j_{\tilde{G}/G} \cdot q_1,$$

where  $q_1$  is the charge of  $A_\mu$  considered as a  $\tilde{G}$  gauge field. This was shown for  $\tilde{G} \simeq SU_2$  in ref. <sup>(10)</sup>, we shall now extend this result to arbitrary simple compact groups.

The proof of (2.6) is most easily given by considering first the case of  $G \simeq SU_2$ . Let  $A_\mu$  be a  $SU_2$  instanton solution with Pontryagin index  $q_1$ . By the embedding  $f$  of  $SU_2$  into  $G \supset SU_2$ , we transform the solution  $A_\mu(x) = A_\mu^\alpha(x) \tilde{X}_\alpha$  into  $A'_\mu(x) = A_\mu^\alpha(x) f(\tilde{X}_\alpha) = A_\mu^\alpha(x) X'_\alpha$  and obtain for the topo-

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<sup>(10)</sup> K. M. BITAR and P. SORBA: *Phys. Rev. D*, **16**, 431 (1977).

logical charge

$$(2.7) \quad q = \frac{1}{16\pi^2} \text{Tr} \int F'_{\mu\nu} \tilde{F}'_{\mu\nu} d^4x = \frac{1}{16\pi^2} [\text{Tr}(X'_3)^2] \int F_{\mu\nu}^{\alpha} \tilde{F}_{\mu\nu}^{\alpha} d^4x,$$

which can also be written, by using (2.5) and the orthonormality of the basis of  $G$ ,  $\text{Tr} X_i X_j = 2\delta_{ij}$ , as

$$(2.8) \quad q = j_I \frac{1}{8\pi^2} \int F_{\mu\nu}^{\alpha} \tilde{F}_{\mu\nu}^{\alpha} d^4x = j_I q_1.$$

Let us consider now the case in which  $\tilde{G}$  is larger than  $SU_2$ . Then eq. (2.7) is still true because the basis of  $G$  is orthonormal. We shall use property ii) mentioned above with  $G_2 = \tilde{G}$  and  $G_3 = G$ . For  $G_1$  we take the  $SU_2$  of index one in  $\tilde{G}$ , and we identify  $X'_3$  with the Cartan generator of this  $SU_2$ . We then find

$$\frac{1}{2} \text{Tr} (X'_3)^2 = j_{su_3|G} = j_{su_3|\tilde{G}} j_{\tilde{G}|G} = j_{\tilde{G}|G},$$

so that (2.6) follows immediately from (2.7).

### 3. — General characterization of embeddings.

Embeddings of gauge fields are defined up to a gauge transformation. This implies that it may be very difficult to recognize embeddings by considering  $A_\mu$  only, since  $A_\mu$  does not transform simply by conjugation under a gauge transformation. The field strength tensor does transform by conjugation, but it does not determine  $A_\mu$  uniquely. Examples of tensors  $F_{\mu\nu}$  to which there corresponds more than one gauge potential (which are not gauge equivalent) are known <sup>(11)</sup>.

A characterization of a gauge field in terms of objects which under gauge transformation transform by conjugation has been given by GU and YANG <sup>(12)</sup>. They prove that for a  $G$  gauge field theory the gauge potential can be determined by the field strength tensor and its gauge derivatives  $D_\rho F_{\mu\nu}$ ,  $D_\rho D_\sigma F_{\mu\nu}$ , ..., up to  $p$ -th order, where

$$(3.1) \quad D_\rho F_{\mu\nu} = \partial_\rho F_{\mu\nu} - i[A_\rho, F_{\mu\nu}]$$

and where the integer  $p$  is at most the order of the group. In fact, for  $SU_2$  and  $SU_3$   $p$  can be taken to be two, in general the smallest possible value of  $p$  is not known.

<sup>(11)</sup> T. T. WU and C. N. YANG: *Phys. Rev. D*, **12**, 3843 (1975).

<sup>(12)</sup> C. H. GU and C. N. YANG: *Scientia Sinica*, **20**, 47 (1977).

In the spirit of the Gu-Yang theorem we can characterize embeddings by the following theorem, which is proved in ref. (2):

*Theorem.* Let  $\mathcal{F}$  be a  $G$  gauge field, where  $G$  is a compact group, on a simply connected open set of  $E^4$  with an analytic gauge potential  $A_\mu(x)$ . If the field strength tensor  $F_{\mu\nu}$  and its covariant derivatives  $D_\rho F_{\mu\nu}$ ,  $D_\rho D_\sigma F_{\mu\nu}$ , ... can be written in terms of a proper subalgebra  $\mathcal{M}$  of  $\mathcal{G}$ , the Lie algebra of  $G$ , then there exists a gauge in which  $A_\mu(x)$  can be written in terms of  $\mathcal{M}$ , i.e. the field  $\mathcal{F}$  is an embedding of the subgroup  $M$  of  $G$  in this open set.

The theorem is general in the sense that it refers to quite arbitrary gauge field configurations, it is not restricted to solutions of the equations of motion. For some applications, like instantons, one is interested in gauge fields defined on the compactified version of the Euclidean space  $S^4$ . To define a gauge field configuration in that case, it is necessary to introduce a patch structure on  $S^4$ , and the global version of the theorem does not follow trivially from the local one given above. In ref. (2) we have shown that the conditions of the theorem stated above are, in fact, sufficient to generalize it to all of  $S^4$ : on  $S^4$  it is, therefore, sufficient to consider  $F$  and its gauge derivatives in one simply connected open region.

For practical purposes this theorem is perhaps not very useful. Its application involves the explicit calculation of many gauge derivatives. In the next section we shall, therefore, give criteria for embeddings which are based on the above theorem, but can be used explicitly for instanton solutions.

#### 4. – Practical criteria for the reducibility of gauge field configurations.

To check the irreducibility of an explicit vector potential the theorem of sect. 3 is not of great practical use. It is, however, possible to formulate criteria in terms of  $F_{\mu\nu}(x)$  which can be used in explicit calculations. In ref. (2) we gave such criteria for the gauge group  $SU_3$ , that is to say we gave necessary and sufficient conditions for the irreducibility of a  $SU_3$  gauge field configuration. This result was subsequently applied to prove that certain  $SU_3$  instanton solutions (5) were indeed irreducible, i.e. were not previously known  $SU_2$  solutions embedded in  $SU_3$ .

These criteria for  $SU_3$  can in principle be generalized to larger groups. However, from our  $SU_3$  result one realizes that any generalization to arbitrary groups must be extremely complicated. Part of this complication arises from the  $U_1$ -subgroups, which are not considered in Dynkin's classification scheme, and are difficult to keep track of. It would, therefore, be useful to exclude  $U_1$ -subgroups, and thus consider only embeddings of semi-simple groups.

Fortunately, the exclusion of  $U_1$ 's corresponds to a situation which is of physical interest. Let us restrict ourselves to analytic gauge field configurations on the compactified Euclidean space  $S^4$ , which satisfy the sourceless Yang-

Mills equations. This eliminates all  $U_1$  gauge groups, since for these groups a gauge field defined on  $S^4$  must have at least one singularity. Thus we can then consider semi-simple algebras only, and in this case we can derive a sufficient condition for the irreducibility of a  $SU_n$  gauge field.

Let  $F(x)$  be any component, or a linear combination of components, of  $F_{\mu\nu}(x)$ . If  $F(x)$  is such that between its eigenvalues there is only one linear relation with  $x$ -independent coefficients (namely  $\text{Tr } F(x) = 0$ ), then the gauge field  $A_\mu$  of  $G \simeq SU_n$  is irreducible.

We can take, without loss of generality,  $F$  diagonal and choose a basis of  $SU_n$  such that the Cartan algebra is represented by diagonal traceless matrices. Suppose that  $A_\mu$  is reducible to a gauge group  $H$ , and that there exist no other linear relations between the eigenvalues of  $F$  than the trace condition. Then  $\mathcal{H}$ , the Lie algebra of  $H$ , contains at least the Cartan algebra of  $\mathcal{G}$ . This can be easily shown by taking linear combinations of matrices  $F(x)$  for different values of  $x$ . Since there are no linear relations besides the trace condition, all  $n - 1$  elements of the Cartan algebra of  $\mathcal{G}$  can be obtained by such linear combinations and are, therefore, in  $\mathcal{H}$ . The rank of  $\mathcal{H}$  is, therefore,  $n - 1$ .

We shall now show that there are no semi-simple proper subalgebras  $\mathcal{H}$  of  $\mathcal{G}$  of rank  $n - 1$ . Let us first consider the case of a simple algebra  $\mathcal{H}$ . Then  $\mathcal{H}$  must be either  $SU_p$ ,  $Sp_{2p}$  or  $O_p$  for some value of  $p$ , or the algebra of an exceptional group. For such algebras it is a simple matter to show, by comparing their dimensions, that, if they are of rank  $n - 1$ , they cannot be proper subalgebras of  $SU_n$ . Therefore,  $\mathcal{H}$  must be nonsimple. If  $\mathcal{H}$  is also regular, it cannot be of rank  $n - 1$ , since any regular subalgebra of  $SU_n$  commutes with at least a  $U_1$  subalgebra. Therefore,  $\mathcal{H}$  is nonregular and nonsimple. For our purposes it is sufficient to regard only maximal subalgebras. For  $SU_n$  all nonsimple, nonregular, maximal subalgebras are known<sup>(1)</sup>, they are  $SU_{n,q} \oplus SU_{q,p}$  with  $n = pq$  and where the subscripts  $q$  and  $p$  indicate the Dynkin indices of  $SU_p$  and  $SU_q$ , respectively. The rank of such a subalgebra is  $p + q - 2$ , while for  $SU_n$  it is  $pq - 1$ . These ranks are equal only for  $p = q = 1$ , which is excluded. Therefore, no semi-simple proper subalgebra  $\mathcal{H}$  of rank  $n - 1$  exists in  $SU_n$ .

This proves that our condition is indeed a sufficient condition for the irreducibility of  $A_\mu$ . This criterion can be applied to  $SU_n$  instantons with cylindrical symmetry which have been obtained by BAIS and WELDON<sup>(6)</sup>. Since their solution is regular, it cannot be a gauge field of a subgroup which contains  $U_1$  factors. Since  $F_{\mu\nu}(x)$  constructed from their solution does not reveal any linear dependences beyond  $\text{Tr } F = 0$ , their solution must be an irreducible  $SU_n$  instanton.

## 5. - Description of embeddings in the parameter space of instantons.

A method to construct all self-dual solutions to the Euclidean Yang-Mills equations for compact Lie groups has been given by ATIYAH *et al.*<sup>(7)</sup>. Their



construction can be formulated in terms of elementary operations on matrices which satisfy certain nonlinear constraints, and whose dimensions depend on the order of the gauge group and the topological charge of the solution. This matrix formulation was made explicit by two groups<sup>(13)</sup>. We shall first present a version of this construction which was given by two of us<sup>(8)</sup> and which emphasizes the geometrical properties of the parameter space (see also ref. (14)). We start from  $SU_n$  instantons and consider all other groups as embeddings in  $SU_n$ .

Let us consider a matrix  $M$ ,  $4k \times 4k$ , positive Hermitian, of rank  $n + 2k$ , which has certain block symmetries:

$$(5.1) \quad M = \begin{bmatrix} \mu & 0 & \sigma & \varrho \\ 0 & \mu - \varrho^+ & \sigma^+ & \\ \sigma^+ - \varrho & \nu & 0 & \\ \varrho^+ & \sigma & 0 & \nu \end{bmatrix}.$$

The complex matrices  $\mu$ ,  $\nu$ ,  $\varrho$  and  $\sigma$  are  $k \times k$  and  $x$ -independent. To parametrize the  $x$ -dependence, we introduce a  $4k \times 2k$  matrix  $\chi$ :

$$(5.2) \quad \chi = \begin{bmatrix} \mathbf{1}_{2k} \\ X \end{bmatrix},$$

where  $X$  is the tensor product of  $\mathbf{1}_k$  with the quaternion  $x$ ,

$$(5.3) \quad x = \begin{bmatrix} x_0 + ix_3 & i(x_1 - ix_2) \\ i(x_1 + ix_2) & x_0 - ix_3 \end{bmatrix}.$$

Now let  $W$  be a  $4k \times n$  matrix satisfying

$$(5.4) \quad W^\dagger M W = \mathbf{1}_n,$$

$$(5.5) \quad W^\dagger M \chi = 0.$$

Then the  $SU_n$  gauge field

$$(5.6) \quad A_\mu = i W^\dagger M \partial_\mu W$$

<sup>(13)</sup> E. F. CORRIGAN, D. B. FAIRLIE, S. TEMPLETON and P. GODDARD: *Nucl. Phys.*, **140** B, 31 (1978); N. H. CHRIST, E. J. WEINBERG and N. K. STANTON: Columbia University preprint (1978).

<sup>(14)</sup> V. G. DRINFELD and YU. I. MANIN: preprint ITEP-72-1978.

gives a self-dual field strength tensor  $F_{\mu\nu}$  corresponding to a topological charge  $k$ . The self-duality is easy to prove by direct calculation. What is difficult is to show that the construction gives all solutions, and that it is essentially unique, *i.e.* to every  $A_\mu$  there corresponds only one matrix  $M$ , up to some simple equivalences (<sup>7</sup>). Clearly, if  $W$  satisfies (5.4) and (5.5), then  $W' = WU(x)$  with  $U(x) \in SU_n$ , is again a solution. This freedom corresponds to gauge transformations of  $A_\mu$ . If we replace everywhere

$$(5.7) \quad \begin{cases} M \rightarrow M' = \Gamma^\dagger M \Gamma, \\ W \rightarrow W' = \Gamma^{-1} W, \end{cases}$$

where

$$(5.8) \quad \Gamma = \begin{bmatrix} K & & & \\ & K & & \\ & & K & \\ & & & K \end{bmatrix}$$

and where  $K$  is an arbitrary invertible  $k \times k$  complex matrix, the pairs  $\{W', M'\}$  and  $\{W, M\}$  will give exactly the same  $A_\mu$ .

Furthermore, we note that we have to require the nonsingularity of the  $2k \times 2k$  matrix  $\chi^\dagger M \chi$  for all  $x$  to avoid singularities of  $A_\mu$ .

Using the transformations (5.7) we can choose in (5.1)

$$(5.9) \quad \mu + \nu = 1/2k,$$

so that  $\text{Tr } M = 1$  and that the remaining equivalences correspond to matrices  $\Gamma$  with  $K = U \in SU_k$ . The space of matrices  $M$  of unit trace, positive Hermitian and with the symmetric structure shown in (5.1) is a compact, convex set  $\mathcal{D}$ . The boundary of  $\mathcal{D}$  is formed by matrices  $M$  of rank less than  $4k$ . In this construction we find all instantons of  $SU_{2k}$  and of  $SU_n$  for  $n > 2k$  (when  $SU_{2k}$  is embedded canonically in  $SU_n$ ), the boundary contains the instantons of  $SU_n$ ,  $n < 2k$ , canonically embedded in  $SU_{2k}$  (<sup>8</sup>).

The parameter space of instantons of charge  $k$  is, therefore, the set  $\mathcal{D}$ , from which are eliminated all  $M$  for which  $\chi^\dagger M \chi$  is singular for some  $x$ , and in which are identified all points which are related by matrices  $\Gamma$  (see eq. (5.7)) with  $K \in SU_k$ . This parameter space is, therefore, the orbit space associated with the action of  $SU_k$  on  $\mathcal{D}$  (minus singular points).

Since this construction gives all instantons of  $SU_{2k}$ , it is clear that also all instantons of subgroups of  $SU_{2k}$  will be obtained. Therefore, the problem of embeddings remains. If we have a matrix  $M$ , satisfying all conditions, how can we be sure to find an  $A_\mu$  which is an irreducible  $SU_n$  solution? Fortunately

there are some simple criteria on the matrices  $\mu, \nu, \varrho$  and  $\sigma$  which characterize the embeddings of the groups  $Sp_{2p}$ ,  $O_m$  and  $SU_p \times SU_q$ . These criteria can be obtained if one starts from the instanton construction for these groups as given in ref. (13), and translates the results in terms of our  $SU_n$  formalism.

Starting from the set  $\mathcal{D}$  we have the following results: the gauge field corresponding to  $M$  is an embedding of

a)  $SU_n$  of Dynkin index one if the rank of  $M$  is  $n + 2k$ .

b)  $Sp_{2p}$  if the rank of  $M$  is  $2k + 2p$  and if on the orbit of  $M$  there is a point for which  $\mu, \nu, \varrho$  and  $\sigma$  are symmetric matrices.

c)  $O_m$  for  $m \geq 4$  if  $k$  is even, if the rank of  $M$  is  $m + 2k$ , and if on the orbit of  $M$  there is a point for which  $\mu, \nu, \varrho$  and  $\sigma$  have the following block structure:

$$\mu, \nu, \varrho, \sigma \sim \left[ \begin{array}{c|c} A & B \\ \hline C & A^t \end{array} \right],$$

where  $B$  and  $C$  are antisymmetric complex matrices,  $A$  is a complex matrix (for  $\mu$  and  $\nu$ , which are Hermitian, of course  $C = B^\dagger$ ). This is the  $O_m$  embedding for which  $A_\mu$  is an antisymmetric Hermitian  $m \times m$  matrix.

d)  $SU_{p,1} \times SU_{q,1}$  (block structure of the Lie algebra; the subscript 1 refers to the Dynkin index) if the rank of  $M$  is  $2k + p + q$ , and if on the orbit of  $M$  there is a point for which  $\mu, \nu, \varrho$  and  $\sigma$  have the block structure

$$\mu, \nu, \varrho, \sigma \sim \left[ \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right],$$

where  $A$  and  $B$ , for each of the matrices  $\mu, \nu, \varrho$  and  $\sigma$ , have dimensions  $k_1 \times k_1$   $k_2 \times k_2$ , respectively, with  $k = k_1 + k_2$ .

Other embeddings, corresponding to higher Dynkin indices, also exist. Their characterization involves more complicated supplementary conditions on  $\mu, \nu, \varrho$  and  $\sigma$ . From the criteria given above we see that, to recognize embeddings in the parameter space of instantons, we have to be able to recognize certain symmetry properties of the matrices  $\mu, \nu, \varrho$  and  $\sigma$ , which may have been distorted by the unitary transformations of type (5.7).

Let us look in more detail at the orbit space associated with the action of  $SU_k$  on  $\mathcal{D}$ . By construction all elements of  $\mathcal{D}$  which are on the same orbit have the same little group (up to a conjugation). We say that two orbits are equivalent if their little groups are conjugated. The orbit space is decomposed

in equivalence classes, which are called strata <sup>(15)</sup>. For example, in the parameter space of instantons the matrix  $M$  with  $\mu, \nu, \varrho$  and  $\sigma$  proportional to  $1_k$  is an orbit, and is in a stratum, stabilized by  $SU_k$  itself. In general, a matrix  $M$  will be stabilized only by the discrete subgroup  $Z_k$  of  $SU_k$ . This can be made more precise by a theorem due to MONTGOMERY and YANG <sup>(15)</sup>: there is a stratum (called the generic stratum) which is open dense (in the topology of the orbit space), and for which the little group is included in all other little groups. In this sense,  $M$  will « almost always » be in the generic stratum and, to be in any other stratum, the matrices  $\mu, \nu, \varrho$  and  $\sigma$  must satisfy supplementary conditions. We shall show the following.

If a matrix  $M$  of rank  $n + 2k$  is not in the generic stratum, the corresponding gauge field configuration will be an embedding of  $SU_{n_1,1} \times SU_{n_2,1}$  for some  $n_1$  and  $n_2$  with  $n = n_1 + n_2$ .

To show this, let us consider a matrix  $M$  which is not in the generic stratum. The little group of  $M$  contains at least a  $U_1$ -subgroup of  $SU_k$ . If we decompose  $\varrho$  and  $\sigma$  in their Hermitian and anti-Hermitian parts, we see that this  $U_1$ -group stabilizes a set of Hermitian matrices  $H_i$ ,  $i = 1, \dots, 5$ , where the  $H_i$  stand for  $\mu, \nu$  and the parts of  $\varrho$  and  $\sigma$ . We can always write  $H_i = 1_k(\text{Tr } H_i)/k + H'_i$ , where  $H'_i$  is traceless and still Hermitian, so that we have

$$[A, H'_i] = 0, \quad i = 1, \dots, 5,$$

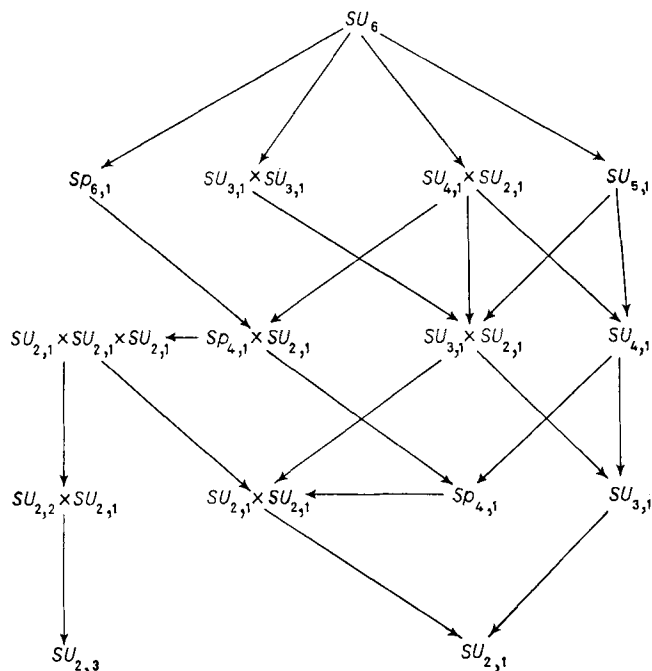
where  $A$  is the generator of the  $U_1$ -group. This means that all  $H'_i$  are in a subalgebra of the Lie algebra of  $SU_k$  which generates the stabilizing group of the element  $A$ . Now, if the group  $SU_k$  acts by conjugation on its Lie algebra, all stabilizers are direct products of  $SU_p$  groups and  $U_1$  factors, and can be put in block form by a  $SU_k$  conjugation. This implies that all  $H_i$  and, therefore,  $\mu, \nu, \varrho$  and  $\sigma$  are in block form and, as we saw above in case *d*), this means that the gauge field is an embedding of a direct product of  $SU_p$ -groups. This shows that, if  $M$  is not in the generic stratum, the gauge field is not an irreducible  $SU_n$  solution. Of course this does not mean that there are no embeddings in the generic stratum.

For small  $k$ , the number of possible embeddings is small, and all possibilities can be considered explicitly. Let us take for instance  $k = 3$ . We then consider all instantons of charge 3 embedded in  $SU_6$ . Since we know (see sect. 2) that the charge is proportional to the Dynkin index, only indices 1 and 3 can occur for simple groups. For semi-simple groups the sum of the indices must not exceed 3. The inclusions of possible embeddings among all the subalgebras

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<sup>(15)</sup> L. MICHEL: in *Statistical Mechanics and Field Theory, Haifa Summer School, 1971*, edited by R. N. SEN and C. WEIL (New York, N. Y., 1972), p. 133.

of  $SU_6$  are given in the following tableau:



From the previous considerations we see that all these embeddings can be characterized by conditions on the parameter space.

### ● RIASSUNTO (\*)

Si discute il problema degli incastonamenti nelle teorie di gauge non abeliane. La (ir)riducibilità di una configurazione di campo di gauge è caratterizzata. Per il caso specifico di soluzioni a istantone, si deriva un criterio pratico per  $SU_n$ . Nella costruzione generale di soluzioni autoduali la riducibilità del campo di gauge implica certe condizioni di simmetria sullo spazio parametrico. Si determinano queste condizioni per incastonamenti  $Sp_n$ ,  $O_n$  e  $SO_p \times SU_q$ .

(\*) Traduzione a cura della Redazione.

### Теоретико-групповые аспекты инстантов.

**Резюме (\*).** — Мы обсуждаем проблему внедрений в неабелевы калибровочные теории. Рассматривается (не)приводимость конфигурации калибровочного поля. Для специального случая инстантных решений мы выводим практический критерий для  $SU_n$ . В общем случае при конструировании самодуальных решений приводимость калибровочного поля подразумевает определенные условия симметрии на пространство параметров. Мы определяем эти условия для включений  $Sp_n$ ,  $O_n$  и  $SU_p \times SU_q$ .

(\*) Переведено редакцией.